JHEP HYPER VERSION

Received: *January 11, 2000*, Accepted: *January 27, 2000*

Supersymmetric 4D orientifolds of type IIA with D6-branes at angles

Ralph Blumenhagen, Lars Görlich and Boris Körs

Humboldt Universität zu Berlin, Institut für Physik
Invalidenstr. 110, 10115 Berlin, Germany
E-mail: blumenha@physik.hu-berlin.de, goerlich@physik.hu-berlin.de, koers@physik.hu-berlin.de

Abstract: We study a certain class of four-dimensional $\mathcal{N} = 1$ supersymmetric orientifolds for which the world-sheet parity transformation is combined with a complex conjugation in the compact directions. We investigate in detail the orientifolds of the $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_6$ and $\mathbb{Z}'_6$ toroidal orbifolds finding solutions to the tadpole cancellation conditions for all models. Generically, all the massless spectra turn out to be non-chiral.

Keywords: D-branes, Superstring Vacua, Supersymmetric Effective Theories.

JHEP01(2000)040

Contents

1. Introduction **1**

2. $\Omega\mathcal{R}$ orientifolds **3**
- 2.1 Proper definition 3
- 2.2 Closed strings 5
- 2.3 Open strings 8

3. The $\mathbb{Z}_4$ orientifold **11**

4. Results for the $\mathbb{Z}_3$ orientifol **15**

5. Results for the $\mathbb{Z}_6$ and $\mathbb{Z}_6'$ orientifolds **16**

6. Conclusions **18**

A. One-loop partition functions **19**
- A.1 Klein bottle 19
- A.2 Annulus 21
- A.3 Möbius strip 22

B. Definitions and modular transformation formulas **23**

1. Introduction

Open string models have largely extended our view on consistent string theory vacua. The foremost promising class of heterotic Calabi-Yau compactifications has become only a small subset of string vacua featuring $\mathcal{N} = 1$ space time supersymmetry in four dimensions and allowing chiral spectra in phenomenological interesting gauge groups. Non-perturbative heterotic vacua containing solitonic five-branes in the background are better described by their dual type-I models, as the heterotic five-brane is mapped to the D5-brane in type I. Generally, type-I models contain D-branes in the background supporting the gauge sector of the low energy theory, whereas gravity propagates in the ten-dimensional bulk. Therefore, such models became quite attractive in recent attempts to establish a unification scenario with the string scale as low as 1 TeV [1].

Type-I models can be described as orientifolds of the type-IIB superstring by the world-sheet parity transformation Ω. Compactification of type-I string theory on various toroidal orbifolds in six and four space dimensions have been studied in a couple of papers [2]–[16]. In particular, for four dimensional compactifications it was found that it is not always possible to satisfy all tadpole cancellation conditions, showing a perturbative inconsistency of some of the models [16].

In [17] we presented a new kind of six-dimensional orientifolds, for which the world-sheet parity transformation is combined with a complex conjugation $\mathcal{R}$ of the internal coordinates. In order to cancel the tadpoles appearing in the Klein bottle amplitude it was necessary to introduce D7-branes intersecting at non-trivial angles into the background. Thus, for the $\mathbb{Z}_3$, $\mathbb{Z}_4$ and $\mathbb{Z}_6$ toroidal orbifolds we found consistent $\Omega\mathcal{R}$ orientifolds in six dimensions leading indeed to anomaly free massless spectra. Generally, the rank of the gauge groups turned out to be reduced by powers of two as compared to the non-compact theory and extra multiplicities appeared in what we called twisted open string sectors. Note, that similar models were also discussed in [18], however the precise relation between our models and the models in [18] is not clear.

In [19] T-duality was used to gain a better understanding of the rank reduction for $\Omega\mathcal{R}$ orientifolds. In contrast to standard Ω orientifolds, it turned out that for $\Omega\mathcal{R}$ orientifolds the background antisymmetric two-form field is a continuous modulus, whereas the off-diagonal part of the internal metric is frozen to discrete values. Turning on these discrete parameters leads to the reduction of the rank of the gauge group by powers of two. In this paper we will present a geometric understanding of the extra multiplicities in twisted open string sectors, which we introduced by hand in [17]. From the loop channel point of view these factors are simply intersection numbers of the D-branes in question.

In the following we will study supersymmetric $\Omega\mathcal{R}$ orientifolds of type IIA in four space-time dimensions. We focus our attention on toroidal orbifolds preserving $\mathcal{N}=1$ supersymmetry and study the $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_6$ and $\mathbb{Z}'_6$ models in detail. Analogous to the six-dimensional case, the Klein bottle tadpoles have to be cancelled by introducing D6-branes at non-trivial angles into the background. Depending on the defining lattices for the three $\mathbb{T}^2$ tori, we find that there exists more than one consistent choice for each $\mathbb{Z}_N$ orbifold. In particular, in contrast to ordinary orientifolds, we find solutions to the tadpole conditions for each $\mathbb{Z}_N$ orbifold. Moreover, for the rank of the resulting gauge groups we always obtain exactly the reduction expected from the general arguments in [19]. At first sight surprisingly, we only get non-chiral massless spectra, where in the two cases, $\mathbb{Z}_4$ and $\mathbb{Z}_6$, accidentally the massless open string spectra fit into $\mathcal{N}=2$ supermultiplets.

This paper is organized as follows. In section 2 we describe the general features of $\Omega\mathcal{R}$ orientifolds in four dimensions. In section 3 we discuss the $\mathbb{Z}_4$ example in some detail, as it already exhibits all the technical steps one has to go through when computing such models. The sections 4 and 5 summarize the results of our computation for the $\mathbb{Z}_3$ and the two $\mathbb{Z}_6$ examples, respectively. In an appendix we have listed the Klein bottle, annulus and Möbius strip amplitudes for all models discussed in this paper. In the conclusions we finish with a brief discussion of non-supersymmetric generalizations of the $\Omega\mathcal{R}$ orientifolds.

2. $\Omega\mathcal{R}$ orientifolds

We consider orientifolds which are obtained by combining the ordinary world sheet parity transformation Ω with a conjugation $\mathcal{R}$ of the three complex coordinates of the six-dimensional torus $\mathbb{T}^6$. Since $\mathcal{R}$ can also be considered as a reflection of three of the six compact coordinates, $\Omega\mathcal{R}$ is rather a symmetry of the type-IIA than of the type-IIB superstring. This operation is accompanied by one of the well known cyclic orbifold groups preserving $\mathcal{N} = 1$ supersymmetry in four space-time dimensions. Thus, the entire orientifold group is given by $\mathbb{Z}_N \cup \Omega\mathcal{R}\mathbb{Z}_N$. Note, that these models are not related by T-duality to standard orientifolds $\mathbb{Z}_N \cup \Omega\mathbb{Z}_N$. Instead they are dual to standard asymmetric orientifolds $\hat{\mathbb{Z}}_N \cup \Omega\hat{\mathbb{Z}}_N$ which we think deserve to be studied in their own right. To begin with we give a more explicit definition of all the elements required.

2.1 Proper definition

The worldsheet parity transformation Ω is combined with a particular reflection $\mathcal{R}$, which in terms of the complex coordinates

$$X_i \equiv x_{10-2i} + i x_{11-2i}\,, \qquad i = 1,2,3\,, \tag{2.1}$$

of a six-dimensional torus can be written as the conjugation

$$\mathcal{R} : X_i \mapsto \bar{X}_i\,. \tag{2.2}$$

In this basis the generator Θ of the orbifold group $\mathbb{Z}_N = \{1, \Theta, \ldots, \Theta^{N-1}\}$ acts diagonally

$$\Theta : X_i \mapsto \exp\left(2\pi i v_i\right) X_i \tag{2.3}$$

and with opposite phases on the conjugate variables. In the same way the complexified fermionic coordinates diagonalize Θ. Both for the RR and the twisted NSNS sector the operation of $\mathcal{R}$ and Θ on degenerate groundstates is given by

$$\begin{aligned} \mathcal{R} &: |s_0, s_1, s_2, s_3\rangle \mapsto |s_0, -s_1, -s_2, -s_3\rangle\,, \\ \Theta &: |s_0, s_1, s_2, s_3\rangle \mapsto \exp\left(2\pi i \vec{v}\cdot\vec{s}\right) |s_0, s_1, s_2, s_3\rangle\,. \end{aligned} \tag{2.4}$$

If the groundstate is not a spinor of the entire light-cone gauge SO(8) little group, but only of a SO($2k$) subgroup, one can formally set the respective s_i in (2.4) to

$\mathbb{Z}_3 : v = (1,1,-2)/3$	$\mathbb{Z}_6' : v = (1,2,-3)/6$	$\mathbb{Z}_8' : v = (1,2,-3)/8$
$\mathbb{Z}_4 : v = (1,1,-2)/4$	$\mathbb{Z}_7 : v = (1,2,-3)/7$	$\mathbb{Z}_{12} : v = (1,4,-5)/12$
$\mathbb{Z}_6 : v = (1,1,-2)/6$	$\mathbb{Z}_8 : v = (1,3,-4)/8$	$\mathbb{Z}_{12}' : v = (1,5,-6)/12$

Table 1: $\mathbb{Z}_N$ groups that preserve $\mathcal{N} = 1$ in $d = 4$.

zero to obtain the correct transformation. There is a subtlety concerning the GSO projection in the twisted sectors. Taking into account that the $\Omega\mathcal{R}$ projection is related to the Ω projection by T-duality in the three directions in which $\mathcal{R}$ acts non-trivially, one is led to the following left moving and right moving world sheet fermion number operators

$$\begin{aligned}(-1)^{F_L} \left| s_0, s_1, s_2, s_3 \right\rangle &= \exp\left(2\pi i (s_0 - s_1 - s_2 - s_3)\right) \left| s_0, s_1, s_2, s_3 \right\rangle , \\ (-1)^{F_R} \left| s_0, s_1, s_2, s_3 \right\rangle &= \exp\left(2\pi i (s_0 + s_1 + s_2 + s_3)\right) \left| s_0, s_1, s_2, s_3 \right\rangle . \end{aligned} \tag{2.5}$$

With this choice of world-sheet fermion number operators the GSO projection is on states satisfying $(-1)^{F_L} = (-1)^{F_R} = -1$. In the untwisted sector of the orientifolds this is apparently equivalent to the usual type-IIA GSO projection and in the twisted sectors it guarantees that $\Omega\mathcal{R}$ is really a symmetry of the type-IIA orbifold theory. In the various open string sectors the GSO projection is always determined by supersymmetry.

Orbifolds that allow $\mathcal{N} = 1$ supersymmetry in four dimensions have been classified in [21, 22] and we display their action on the complex basis in terms of the v_i in table 1.

In this paper we will restrict ourselves to explicit computations for the orbifolds $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_6$ and $\mathbb{Z}_6'$. It has been established for ordinary four dimensional type-I vacua that there exists no solution to the tadpole cancellation conditions for the cases $\mathbb{Z}_4$, $\mathbb{Z}_8$, $\mathbb{Z}_8'$ and $\mathbb{Z}_{12}$ [14].[1] In contrast to that, we obtain perturbatively consistent $\Omega\mathcal{R}$ orientifolds for any $\mathbb{Z}_N$ model studied in this paper.

For the compact $\mathbb{T}^6$ one needs to use a lattice that allows a crystallographic action of the above cyclic groups. Up to overall scales in each $\mathbb{T}^2$ factor, for the $\mathbb{Z}_3$ and $\mathbb{Z}_6$ case we choose the root lattice of $\mathrm{SU}(3)^3$ and for the $\mathbb{Z}_4$ case the root lattice of $\mathrm{SU}(2)^6$. Except for the $\mathbb{Z}_3$ example these are not of the type explored in [23], which allow to take the coxeter element of the Lie algebra as the generator of the orbifold group. As the particular choice of the lattice enters at various important points into the calculation of the tadpole cancellation conditions and the spectrum, different lattices may lead to even more inequivalent models. With our choice we obtain twelve inequivalent models from the four orbifolds by the freedom of choice in the relative orientations of the lattices with respect to the reflection $\mathcal{R}$.

[1] See also [20] for a recent discussion of similar models leading to supersymmetry breaking on the D-branes.

In the following we describe in some detail the general features that arise in computing the three contributions to the massless tadpoles arising in the Klein bottle, annulus and Möbius strip amplitudes. The truly technical details of the explicit expressions can be found in the appendix. Part of the following discussion already appeared in [17], but in a couple of important questions substantial progress could be achieved. For instance, we clarify the appearance of extra weight factors for twisted open string sectors in the annulus and Möbius strip amplitudes. In [17] these factors were detected by employing the equivalence between the tree channel and the loop channel description of the annulus. Now, we can give a striking geometric interpretation of these factors as intersection numbers of the D-branes supporting the twisted open string sector.

2.2 Closed strings

In the closed string sector the states with excitations of left-moving oscillators combine with their complex conjugate images on the right-moving side into $\Omega\mathcal{R}$ invariant states

$$\begin{aligned}(\Omega\mathcal{R})\left(\alpha_{n+kv_i}\tilde{\overline{\alpha}}_{n+kv_i}\right)(\Omega\mathcal{R})^{-1} &= \left(\alpha_{n+kv_i}\tilde{\overline{\alpha}}_{n+kv_i}\right),\\ (\Omega\mathcal{R})\left(\overline{\alpha}_{n-kv_i}\tilde{\alpha}_{n-kv_i}\right)(\Omega\mathcal{R})^{-1} &= \left(\overline{\alpha}_{n-kv_i}\tilde{\alpha}_{n-kv_i}\right),\end{aligned} \tag{2.6}$$

with α representing any kind of bosonic or fermionic ladder operator of the k-twisted sector and $n \in \mathbb{Z}$ or $n \in \mathbb{Z}+1/2$. As opposed to standard orientifolds the $\Omega\mathcal{R}$ invariant states in (2.6) are also invariant under the $\mathbb{Z}_N$ action

$$\begin{aligned}\Theta\left(\alpha_{n+kv_i}\tilde{\overline{\alpha}}_{n+kv_i}\right)\Theta^{-1} &= \left(\alpha_{n+kv_i}\tilde{\overline{\alpha}}_{n+kv_i}\right),\\ \Theta\left(\overline{\alpha}_{n-kv_i}\tilde{\alpha}_{n-kv_i}\right)\Theta^{-1} &= \left(\overline{\alpha}_{n-kv_i}\tilde{\alpha}_{n-kv_i}\right).\end{aligned} \tag{2.7}$$

The important point to notice is that $\Omega\mathcal{R}$ does not exchange a Θ^k twisted sector with a Θ^{N-k} twisted sector, so that all twisted sectors lead to a non-vanishing contribution to the loop channel Klein bottle amplitude

$$\mathcal{K} = 4c\int_0^\infty \frac{dt}{t^3}\ \mathrm{Tr}_{\mathrm{U+T}}\left(\frac{\Omega\mathcal{R}}{2}\frac{\left(1+\Theta+\cdots+\Theta^{N-1}\right)}{N}\frac{\left(1+(-1)^F\right)}{2}e^{-2\pi t\left(L_0+\bar{L}_0\right)}\right), \tag{2.8}$$

with $c \equiv V_4/\left(8\pi^2\alpha'\right)^2$ and the momentum integration in the non-compact space-time already performed. Remember, that for conventional orientifolds only the untwisted and, if present, the $\mathbb{Z}_2$ twisted sectors contribute to this amplitude. Here, the relation

$$\Omega\mathcal{R}\Theta^k = \Theta^{N-k}\Omega\mathcal{R} \tag{2.9}$$

implies that in the tree channel only untwisted closed string states propagate between the two cross-caps on both sides of the tube. Thus, world sheet consistency requires that transforming the loop channel amplitude (2.8) into tree channel must lead to an amplitude in which only $\mathbb{Z}_N$ invariant states from the untwisted sector contribute. In doing the computation one realizes that this world-sheet consistency condition is not always automatically satisfied. Therefore, the completion of the $\mathbb{Z}_N$ projector in the tree channel will serve as the guiding principle in constructing consistent models.

Since the action of Θ in the basis of $\Omega\mathcal{R}$ invariant states is always trivial on the oscillator part, in each $\mathbb{Z}_N$ twisted sector the complete Klein bottle amplitude factorizes into a trace over the oscillators times a trace over the lattice. To compute the lattice part, one needs to determine the winding (W) and the Kaluza-Klein (KK) modes which are invariant under the operator $\Omega\mathcal{R}\Theta^k$ appearing in the trace. To begin with, let us consider a single two-dimensional torus, on which Θ acts as a rotation by an angle $\phi = 2\pi/N$. In fact the entire lattice partition function of the six-dimensional torus factorizes into a product of three two-dimensional tori with an action of Θ as a $2\pi v_i/N$ rotation on each individual factor. Let us define the lattice type **A** to be the orientation of the $\mathbb{Z}_N$ lattice such that the reflection $\mathcal{R}$ acts orthogonally to one of the two basis vectors that span the lattice. In this case the relation

$$\left(\Omega\mathcal{R}\Theta^k\right)\Theta^2 = \Theta^{-1}\left(\Omega\mathcal{R}\Theta^k\right)\Theta \tag{2.10}$$

implies that the partition function of the lattice with $\Omega\mathcal{R}\Theta^k$ insertion does only depend on k being even or odd. Rotating the lattice by an angle of π/N for even and $\pi/(2N)$ for odd N leads to the lattice type **B** and constitutes the only non-trivial rotation maintaining a crystallographic action of $\mathcal{R}$. Due to the relation

$$\Theta^{-1/2}\left(\Omega\mathcal{R}\Theta^k\right)\Theta^{1/2} = \left(\Omega\mathcal{R}\Theta^k\right)\Theta \tag{2.11}$$

the two cases, even and odd, get exchanged. Note, that in the special case of the $\mathbb{Z}_3$ orbifold the rotation $\Theta^{1/2}$ is a symmetry of the lattice implying that the Klein bottle contributions are entirely independent of k.

It appears to be the correct choice for world-sheet consistency, that whenever the orbifold model has two complex directions with Nv_i odd and only one with Nv_i even, $\mathcal{R}$ acts differently on the two odd directions and arbitrarily on the even direction. Said differently, we choose the lattices **AB** or **BA** for the two odd directions and in the even direction we are free to choose either **A** or **B**. In this way one produces a partition function which in the untwisted sector loop channel yields identical contributions for all Θ^k insertions in the trace. Accidentally, this remains true for the twisted sectors with the exception of the $\mathbb{Z}_6'$ orientifold. In the $\mathbb{Z}_6'$ case, the two choices **AB** and **BA** for the two odd tori lead to different models. Summarizing, for the $\mathbb{Z}_3$ orbifold we have the four inequivalent choices $\{\mathbf{AAA}, \mathbf{AAB}, \mathbf{ABB}, \mathbf{BBB}\}$, for the $\mathbb{Z}_4$ and the $\mathbb{Z}_6$ the two choices $\{\mathbf{ABA}, \mathbf{ABB}\}$ and finally for the $\mathbb{Z}_6'$ the four choices $\{\mathbf{AAB}, \mathbf{ABB}, \mathbf{BAA}, \mathbf{BBA}\}$.

We have also inspected some of the cases without any relative angle between the lattices of the two-dimensional tori, but found that these never lead to the complete

$\mathbb{Z}_N$ projector in the tree channel. Thus, the world-sheet consistency condition is violated and in the spirit of [16] it is tempting to speculate that this might hint to the existence of non-perturbative states which remove this discrepancy. To prove these speculation it is of primary importance to find a heterotic or F-theory dual description of the $\Omega\mathcal{R}$ orientifolds.

In order to compute the loop channel Klein bottle amplitude in the Θ^l twisted sector, we have to know which of the Θ^l fixed points are invariant under the action of $\Omega\mathcal{R}\Theta^k$. These multiplicities arise from the center of mass coordinate of the string. It is required to be invariant under the twisting element as well as under the insertion in the trace. With the exception of the $\mathbb{Z}_6'$ orientifold the number of invariant fixed points does not depend on k, though different individual points are invariant. In the $\mathbb{Z}_6'$ example the Θ^2 and Θ^4 twisted sectors split into a sum of two terms containing the $\Omega\mathcal{R}\Theta^{2k}$ and $\Omega\mathcal{R}\Theta^{2k+1}$ insertions in the trace, respectively. Note, that the fixed point multiplicities non-trivially conspire with the relative factors arising from the modular transformation of the lattice contributions in order to complete the tree channel projector.

Regarding these subtleties the computation of the various amplitudes is a straightforward though still tedious task. All the results for the different traces involved can be found in the appendix and the $\mathbb{Z}_4$ example will be discussed in greater detail in the following chapter.

Finally, we need to compute the massless spectra. The first step is to add up the ground state energies in the twisted sectors using the general formula

$$E_0 = \sum_{i=0}^{3} \left(\frac{1}{24} - \frac{1}{8} \left(2kv_i - 1\right)^2 \right), \tag{2.12}$$

which is presented for a complex boson of the Θ^k-twisted sector. For the fermions the sign changes to $-E_0$ and in the NSNS sector one needs to replace v_i by $v_i + 1/2$. In the untwisted sector Θ invariant left- and right-moving massless states have to be symmetrized and antisymmetrized under $\Omega\mathcal{R}$ in the NSNS and the RR sector respectively. This always contributes the graviton and dilaton from the NSNS sector and some model dependent number of additional neutral chiral multiplets. For the twisted sectors one carefully needs to inspect the transformation properties of the fixed points. For instance those which are invariant under $\mathcal{R}$ and Θ need to be properly symmetrized and antisymmetrized as in the untwisted sector. It also happens that the fixed points are only invariant under a combination of $\mathcal{R}$ and Θ or even under neither of the two transformations. In the latter case no symmetrization at all has to take place. Following this procedure, the total number of chiral (C) and vector (V) multiplets[2] is exactly the sum of the Hodge numbers of the blown up

[2]We would like to acknowledge that [24] has drawn our attention to the necessity of further distinguishing states in the twisted RR sectors, that transform as scalars respectively as vectors.

Orbifold group	Model	untwisted	$\Theta+\Theta^{-1}$ twisted	$\Theta^2\,(+\Theta^{-2})$ twisted	Θ^3 twisted
$\mathbb{Z}_3$	**AAA**	9C	14C+13V	absent	absent
$\mathbb{Z}_3$	**AAB**	9C	15C+12V	absent	absent
$\mathbb{Z}_3$	**ABB**	9C	18C+9V	absent	absent
$\mathbb{Z}_3$	**BBB**	9C	27C	absent	absent
$\mathbb{Z}_4$	**ABA**	6C	16C	15C+1V	absent
$\mathbb{Z}_4$	**ABB**	6C	12C+4V	15C+1V	absent
$\mathbb{Z}_6$	**ABA**	5C	2C+1V	9C+6V	10C+1V
$\mathbb{Z}_6$	**ABB**	5C	3C	12C+3V	10C+1V
$\mathbb{Z}'_6$	**AAB**	4C	7C+5V	14C+4V	10C+2V
$\mathbb{Z}'_6$	**ABB**	4C	9C+3V	18C	10C+2V

Table 2: Closed string spectra.

toroidal orbifold

$$\text{Number of neutral multiplets C+V } = h^{1,1} + h^{2,1}. \tag{2.13}$$

In particular, for inequivalent orientifold models with identical orbifold groups we find the same net number of multiplets in the closed string spectrum, while the individual states are different. In table 2 we display the closed string spectra for the orientifolds discussed in this paper.

In the untwisted sector there are always the graviton and the dilaton multiplet in addition to the chiral multiplets as given in the table.

2.3 Open strings

Analogously to the six-dimensional models studied in [17], in order to cancel the tadpoles from the Klein bottle we have to introduce D6-branes intersecting each other at non-trivial angles. They always stretch in one direction on each two-dimensional torus and need to intersect in a Θ and $\mathcal{R}$ invariant fashion. Matching the modings of the fields in the closed string twisted sectors, open strings stretching between D-branes at relative angle πv_i carry fields with modings shifted by v_i. Thus, we are led to consider arrays of N kinds of D-branes at this relative angle. On each torus $\mathbb{T}^2$ one of them is located in the fixed plane of the reflection $\mathcal{R}$ and the remaining ones are obtained by successively applying the rotation $\Theta^{1/2}$. Note that some of them will coincide in some complex directions eventually.

In the open string sector one has to compute the annulus

$$\mathcal{A} = c\int_0^\infty \frac{dt}{t^3}\,\mathrm{Tr}_{\mathrm{open}}\left(\frac{1}{2}\frac{1+\Theta+\cdots+\Theta^{N-1}}{N}\frac{1+(-1)^F}{2}e^{-2\pi t L_0}\right) \tag{2.14}$$

and the Möbius strip amplitude

$$\mathcal{M} = c \int_0^\infty \frac{dt}{t^3}\, \mathrm{Tr}_{\rm open} \left(\frac{\Omega\mathcal{R}}{2} \frac{1+\Theta+\cdots+\Theta^{N-1}}{N} \frac{1+(-1)^F}{2} e^{-2\pi t L_0} \right). \qquad (2.15)$$

These amplitudes only receive a non-vanishing contribution from open strings in the $(6_i, 6_{i+n})$ sector, if the action of the operator in the trace leaves the two D6-branes and the orientation of the open string invariant. If the moding of the fields in the $(6_i, 6_{i+n})$ open string sector is similar to the moding of the fields in the Θ^k twisted closed string sector, we also use the term "Θ^k twisted" for the open string sector. For even N the operator $\Theta^{N/2}$ leaves all D6_i-branes invariant and has a non-trivial action on the Chan-Paton factors as usually described by a matrix $\gamma^{(i)}_{N/2}$. Correspondingly, the annulus amplitude gives rise to an additional $\mathbb{Z}_2$ twisted sector tadpole.

Some care has to be taken when computing the contributions of momenta and winding states, which are present for $(6_i, 6_{i+n})$ strings whenever the two branes coincide in some complex direction and the operator in the trace acts trivially there. One then needs to consider the orientation of the brane with respect to the lattice and determine the allowed winding and momentum states. In the Möbius strip amplitude these KK and W modes also have to be invariant under $\Omega\mathcal{R}$. Generally this leads to a doubling of winding states as compared to the annulus, with the exception of the **A** type SU$(2)^2$ lattice. The present choice of lattices and branes guarantees that the contributions are independent of the operator in the trace for a given sector, except again for the $(6_i, 6_{i+2})$ strings in the $\mathbb{Z}_6'$.

A very important point in the computation are the extra multiplicities of some twisted open string sectors introduced in [17] by hand in order to satisfy the worldsheet consistency condition. These extra multiplicities were determined by the complete projector in the tree channel and nicely led to tadpole cancellation and anomaly free massless spectra. Unfortunately, in [17] we could not present an understanding of these factors from the loop channel point of view, which we would like to do now. The source for the extra multiplicities is again the center of mass coordinate of the open string, which is required to be an intersection point of the two D-branes. The number of such intersection points of the respective D-branes in question perfectly reproduces these extra factors and thus gives a striking geometrical interpretation. For computing the annulus and Möbius strip amplitude, these intersection points need to be invariant under the operator in the trace. It turns out that the number of invariant intersection points of $(6_i, 6_{i+n})$ branes is independent of i with the exception of $n = 2$ in the $\mathbb{Z}_6'$ orientifold. In [19] these extra multiplicities have been related via T-duality to extra factors for twisted sectors in ordinary Ω orientifolds with background $B^{\mu\nu}$ field. So far we have presented all the novel ingredients needed to compute the two open string amplitudes and the details are left to the appendix and to the discussion of the $\mathbb{Z}_4$ example.

Let us now discuss the main steps for computing the massless open string spectrum. One has to be very careful again with the contributions of the different intersection points. First of all we notice, that one always gets a tadpole cancellation condition of the form

$$(\mathrm{M} - 2^{\kappa})^2 = 0 \tag{2.16}$$

fixing the number M of D6-branes of each type. For even $\mathbb{Z}_N$ there is an additional $\mathbb{Z}_2$ twisted tadpole condition, which requires

$$\mathrm{tr}\left(\gamma^{(i)}_{N/2}\right) = 0 \tag{2.17}$$

exactly resembling the computation for the standard six-dimensional $\mathbb{Z}_2$ orientifold discussed by Gimon and Polchinski (GP) in [4]. We can therefore copy their solution for the Chan-Paton degrees of freedom, which implies an SO(M) gauge group on each stack of branes, broken to its U(M/2) subgroup, if N is even. We simplify the analysis by looking at those open strings only, which begin on one of the branes located in the fixed plane of $\mathcal{R}$, as all other states are related to these by some action of the orientifold group. Again, for the $\mathbb{Z}'_6$ model the situation is slightly different and one has to consider the branes of odd and even i separately.

For the massless states in the untwisted NS sector the action of the ordinary Ω on D9-branes is identical to the action of $\Omega\mathcal{R}$ on D6-branes, as the additional signs which Ω contributes to the Dirichlet directions just cancel the reflection. For even N we therefore only need to distinguish further, whether $\Theta^{N/2}$ acts by a reflection or trivially on a given state, i.e. if $Nv_i/2$ is integer or not. A single physical state with a reflection contributes two states of a multiplet in the antisymmetric representation of the gauge group, while a state with trivial action contributes a single state of a multiplet in the adjoint representation. By inspecting table 1 one can easily realize, that generically there is one complex direction, which $\Theta^{N/2}$ acts trivially on, while it reflects the other two. By this reasoning we find, that the untwisted sector always contributes a vector and a chiral multiplet in the adjoint representation of the gauge group from the trivial directions and four chiral multiplets in the antisymmetric representation from the reflected directions. For odd N, in addition to the vector multiplet there simply appear 3 chiral multiplets in the antisymmetric representation.

Similar to the closed string case, for the twisted sectors things become more involved. First one needs to do the same distinction concerning the Chan-Paton labels as for the untwisted sector above. Moreover, one has to distinguish the contributions of the various intersection points, which may be invariant under Θ and $\mathcal{R}$ or not. Finally, one must keep track of various phase factors, that appear via modular transformation and provide additional relative signs. This needs to be taken into account, when doing any symmetrization or antisymmetrization of Chan-Paton labels. Since the computation of the open string spectrum of the four-dimensional $\Omega\mathcal{R}$ orientifolds is similar to a $\mathbb{Z}_2$ standard orientifold, we only find non-chiral spectra in

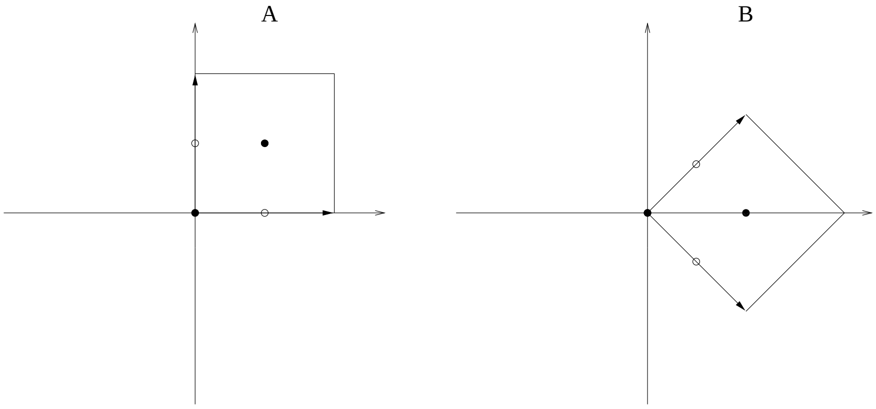

Figure 1: The $\mathbb{Z}_4$ lattices.

mostly rather small gauge groups. Remember, that standard four-dimensional $\mathbb{Z}_N$ orientifolds for $N > 2$ generically have chiral spectra.

After having explained all the principles, we now come to the detailed discussion of one example. We have chosen the $\mathbb{Z}_4$ as it already exhibits all the generic features we have went through while not being as complicated in technical terms as the $\mathbb{Z}_6'$.

3. The $\mathbb{Z}_4$ orientifold

The $\mathbb{Z}_4$ action is given by $v = (1, 1, -2)/4$ and up to scales for the $\mathbb{T}^6$ we choose the root lattice of $\mathrm{SU}(2)^6$. In figure 1 we depicted the two possible choices of relative orientations of the $\mathrm{SU}(2)^2$ lattice with respect to the fixed line of $\mathcal{R}$, which is identical to one of the coordinate axes for each $\mathbb{T}^2$.

The KK and W states invariant under $\Omega\mathcal{R}\Theta^{2k}$ are

$$\begin{aligned} p^{\mathbf{A}} &= \frac{m}{R}\,, \qquad & l^{\mathbf{A}} &= nR\,, \\ p^{\mathbf{B}} &= \sqrt{2}\frac{m}{R}\,, \qquad & l^{\mathbf{B}} &= \sqrt{2}nR\,, \end{aligned} \tag{3.1}$$

for the respective orientations of the lattice. The states invariant under $\Omega\mathcal{R}\Theta^{2k+1}$ are obtained by exchanging **A** and **B** in (3.1) for the first two $\mathrm{SU}(2)^2$ tori, whereas they are identical to (3.1) for the third $\mathbb{T}^2$. By world-sheet consistency we are forced to choose the lattices in the first two $\mathbb{T}^2$ tori as **AB** and in the third $\mathbb{T}^2$ we are free to choose it either **A** or **B**. Thus, we get the two inequivalent models **ABA** and **ABB**. We display all results first for the **ABA** model and state relative factors for the **ABB** case separately. The number of fixed points in Θ and Θ^3 twisted sectors is 16 in either case, but for **ABB** only 8 are invariant under $\Omega\mathcal{R}\Theta^k$. In the Θ^2 twisted sector one has also 16 fixed points, of which one half is invariant under $\Omega\mathcal{R}\Theta^k$ both for **ABA** and **ABB**. Note, that the expected relative prefactors for the terms in the complete projector in the tree channel amplitude are

$$\begin{aligned}
\Theta^1: &\ \left(-2\sin\left(\frac{\pi}{4}\right)\right)^2\left(-2\sin\left(-\frac{\pi}{2}\right)\right)=4\,,\\
\Theta^2: &\ \left(-2\sin\left(\frac{\pi}{2}\right)\right)\left(-2\sin\left(-\frac{\pi}{2}\right)\right)=-4\,,\\
\Theta^3: &\ \left(-2\sin\left(\frac{3\pi}{4}\right)\right)^2\left(-2\sin\left(-\frac{3\pi}{2}\right)\right)=-4\,.
\end{aligned} \tag{3.2}$$

as explained in the appendix. This will be found to be consistent for both **ABA** and **ABB**.

After this fixed point analysis we have all the ingredients needed to compute the Klein bottle trace

$$\begin{aligned}
\mathcal{K}=c\int_0^\infty \frac{dt}{t^3}\Big(&\mathcal{K}^{(0)}_{\left(\frac{1}{4},\frac{1}{4},\frac{1}{2}\right)}\,\mathcal{L}\,[1,1]^2\,\mathcal{L}\,[2,2]\,+\\
&+16\,\mathcal{K}^{(1)}_{\left(\frac{1}{4},\frac{1}{4},-\frac{1}{2}\right)}+8\mathcal{K}^{(2)}_{\left(\frac{1}{4},-\frac{1}{4},0\right)}\,\mathcal{L}\,[1,1]+16\mathcal{K}^{(3)}_{\left(\frac{1}{4},\frac{1}{4},-\frac{1}{2}\right)}\Big)\,,
\end{aligned} \tag{3.3}$$

using the notations defined in the appendix. Numerically, all the twisted sector contributions vanish, but we have spelled them out to demonstrate the formal appearance of the projector in the tree channel. The modular S transformation leads to

$$\begin{aligned}
\tilde{\mathcal{K}}=32c\int_0^\infty dl\,\Big(&\tilde{\mathcal{K}}^{(0)}_{\left(\frac{1}{4},\frac{1}{4},-\frac{1}{2}\right)}\,\tilde{\mathcal{L}}\,[4,4]^2\,\tilde{\mathcal{L}}\,[2,2]\,+\\
&+4\,\tilde{\mathcal{K}}^{(1)}_{\left(\frac{1}{4},\frac{1}{4},-\frac{1}{2}\right)}-4\tilde{\mathcal{K}}^{(2)}_{\left(\frac{1}{4},-\frac{1}{4},0\right)}\,\tilde{\mathcal{L}}\,[4,4]-4\tilde{\mathcal{K}}^{(3)}_{\left(\frac{1}{4},\frac{1}{4},-\frac{1}{2}\right)}\Big)\,.
\end{aligned} \tag{3.4}$$

The relative factors are in perfect match with the expected ones in (3.2). This is also true for the **ABB** lattice, as the only difference is an overall factor of $1/2$, originating both from the lattice partition functions and from the reduction of the number of invariant fixed points. This is quite a remarkable coincidence, indeed.

To cancel the tadpole of the Klein bottle, we introduce $\mathrm{D6}_i$-branes into the background as shown in figure 2, where $i=1,\ldots,4 \bmod 4$. The relative angles are πv_i and the 6_1-branes are lying entirely inside the fixed plane of $\mathcal{R}$.

As the orientifold group does not mix even and odd numbered branes, they will carry distinct factors of the gauge group and the only fields that transform non-trivially under both factors arise in the $(6_i,6_{i+1})$ and $(6_i,6_{i+3})$ open string sectors.

As we have already anticipated in section 2, putting for instance the D6-branes in the third $\mathbb{T}^2$ at an angle $\phi=\pi/4$ relative to the x_4 axes we would get both different KK and W contributions and different intersection numbers. The transformation to tree channel shows that for this choice of the D6-branes there is no chance to cancel tadpoles. Thus, tadpole cancellation uniquely fixes the location of the D6-branes.

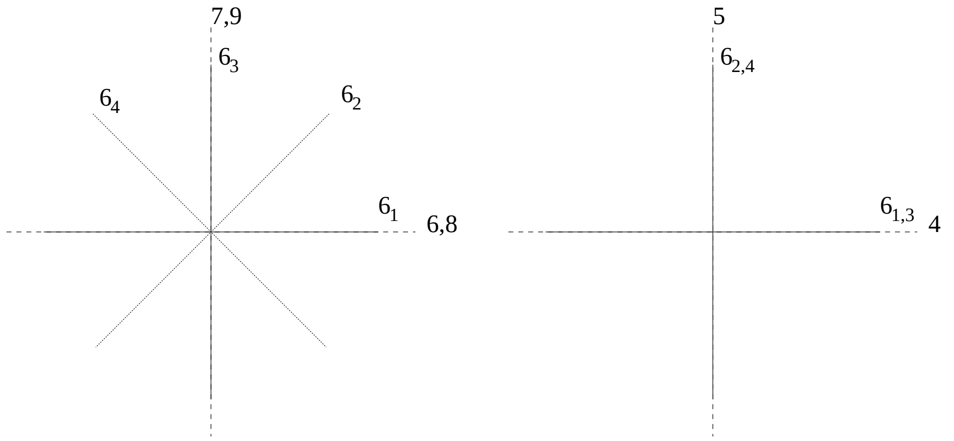

Figure 2: Branes on the $\mathbb{Z}_4$ torus.

For the correct location of the D6-branes shown in figure 2 the complete annulus amplitude reads

$$\begin{aligned}\mathcal{A} = \frac{c}{4}\int_0^\infty \frac{dt}{t^3}\Bigg(& \mathrm{M}^2\mathcal{A}^{(0,0)}_{\left(\frac14,\frac14,-\frac12\right)}\,,\mathcal{L}\,[2,2]^2\,\mathcal{L}\,[1,1] + \frac14\left(\sum_{i=1}^4 \gamma_i^2\right)\mathcal{A}^{(0,2)}_{\left(\frac14,-\frac14,0\right)}\,\mathcal{L}\,[2,2]+\\ &+\mathrm{M}^2\mathcal{A}^{(1,0)}_{\left(\frac14,\frac14,-\frac12\right)} + \frac14\left(\sum_{i=1}^4 \gamma_i\gamma_{i+1}\right)\mathcal{A}^{(1,2)}_{\left(\frac14,\frac14,-\frac12\right)} + 2\mathrm{M}^2\mathcal{A}^{(2,0)}_{\left(\frac14,-\frac14,0\right)}\,\mathcal{L}\,[2,2]+\\ &+\left(\gamma_1\gamma_3+\gamma_2\gamma_4\right)\mathcal{A}^{(2,2)}_{\left(\frac14,-\frac14,0\right)}\,\mathcal{L}\,[2,2] + \mathrm{M}^2\mathcal{A}^{(3,0)}_{\left(\frac14,\frac14,-\frac12\right)} +\\ &+\frac14\left(\sum_{i=1}^4 \gamma_i\gamma_{i+1}\right)\mathcal{A}^{(3,2)}_{\left(\frac14,\frac14,-\frac12\right)}\Bigg)\,, \end{aligned}\tag{3.5}$$

where we have introduced the notation $\gamma_i = \mathrm{tr}(\gamma_2^{(i)})$ for the action of the $\mathbb{Z}_2$ element on the Chan-Paton factors for the four different kinds of D6_i-branes. The amplitude (3.5) transforms to the tree channel expression:

$$\begin{aligned}\tilde{\mathcal{A}} = \frac{c}{8}\int_0^\infty dl\,\Bigg(& \mathrm{M}^2\tilde{\mathcal{A}}^{(0,0)}_{\left(\frac14,\frac14,-\frac12\right)}\,\tilde{\mathcal{L}}\,[1,1]^2\,\tilde{\mathcal{L}}\,[2,2] + 2\left(\sum_{i=1}^4 \gamma_i^2\right)\tilde{\mathcal{A}}^{(0,2)}_{\left(\frac14,-\frac14,0\right)}\,\tilde{\mathcal{L}}\,[1,1]+\\ &+4\mathrm{M}^2\tilde{\mathcal{A}}^{(1,0)}_{\left(\frac14,\frac14,-\frac12\right)} + \left(\sum_{i=1}^4 \gamma_i\gamma_{i+1}\right)\tilde{\mathcal{A}}^{(1,2)}_{\left(\frac14,\frac14,-\frac12\right)} - 4\mathrm{M}^2\tilde{\mathcal{A}}^{(2,0)}_{\left(\frac14,-\frac14,0\right)}\,\tilde{\mathcal{L}}\,[1,1]+\\ &+\,2\left(\gamma_1\gamma_3+\gamma_2\gamma_4\right)\tilde{\mathcal{A}}^{(2,2)}_{\left(\frac14,-\frac14,0\right)}\,\tilde{\mathcal{L}}\,[1,1] - 4\mathrm{M}^2\tilde{\mathcal{A}}^{(3,0)}_{\left(\frac14,\frac14,-\frac12\right)} -\\ &-\left(\sum_{i=1}^4 \gamma_i\gamma_{i+1}\right)\tilde{\mathcal{A}}^{(3,2)}_{\left(\frac14,\frac14,-\frac12\right)}\Bigg)\,. \end{aligned}\tag{3.6}$$

If we change the lattice to **ABB**, while keeping the branes fixed, we get an extra overall factor of 2 due to different KK and W sums as well as a different intersection number for $(6_i, 6_{i+1})$ strings. In both cases the projector in the tree channel is complete.

The final contribution comes from the Möbius strip amplitude, where the invariant windings and momenta are independent of the particular choice of the lattice:

$$\begin{aligned}\mathcal{M} = -\frac{c\,\mathrm{M}}{4}\int_0^\infty \frac{dt}{t^3}\,\Big(&\mathcal{M}^{(0,0)}_{\left(\frac{1}{4},\frac{1}{4},-\frac{1}{2}\right)}\,\mathcal{L}\,[2,2]^2\,\mathcal{L}\,[1,4] - 4\mathcal{M}^{(2,1)}_{\left(\frac{1}{4},\frac{1}{4},-\frac{1}{2}\right)} + \\ &+\mathcal{M}^{(0,2)}_{\left(\frac{1}{4},-\frac{1}{4},0\right)}\,\mathcal{L}\,[2,2] + 4\mathcal{M}^{(2,3)}_{\left(\frac{1}{4},\frac{1}{4},-\frac{1}{2}\right)}\Big)\,. \end{aligned} \tag{3.7}$$

This leads to the tree channel expression

$$\begin{aligned}\tilde{\mathcal{M}} = -4c\,\mathrm{M}\int_0^\infty dl\,\Big(&\tilde{\mathcal{M}}^{(0)}_{\left(\frac{1}{4},\frac{1}{4},-\frac{1}{2}\right)}\,\tilde{\mathcal{L}}\,[4,4]^2\,\tilde{\mathcal{L}}\,[8,2] - 4\tilde{\mathcal{M}}^{(3)}_{\left(\frac{1}{4},\frac{1}{4},-\frac{1}{2}\right)} - \\ &- 4\tilde{\mathcal{M}}^{(2)}_{\left(\frac{1}{4},-\frac{1}{4},0\right)}\,\tilde{\mathcal{L}}\,[4,4] + 4\tilde{\mathcal{M}}^{(1)}_{\left(\frac{1}{4},\frac{1}{4},-\frac{1}{2}\right)}\Big)\,. \end{aligned} \tag{3.8}$$

There are no extra factors whatsoever in the Möbius strip amplitude when switching to the **ABB** lattice. Summarizing, we get the following untwisted and $\mathbb{Z}_2$ twisted tadpole cancellation conditions for the **ABA** model

$$\begin{aligned}\frac{1}{8}\left(\mathrm{M}^2 - 32\mathrm{M} + 256\right) &= \frac{1}{8}\left(\mathrm{M} - 16\right)^2 = 0\,,\\ \gamma_1^2 + \gamma_3^2 + \gamma_1\gamma_3 &= 0\,,\\ \gamma_2^2 + \gamma_4^2 + \gamma_2\gamma_4 &= 0\,. \end{aligned} \tag{3.9}$$

Deriving the $\mathbb{Z}_2$ twisted tadpole cancellation conditions one realizes that the $6_{1,3}$ and $6_{2,4}$ branes are charged under different twisted RR 5-forms. This can be seen by taking all the volume factors appearing in the amplitude into account individually. More precisely, the twisted sector tadpole condition receives contributions from the sixteen individual $\mathbb{Z}_2$ fixed points. Analyzing the intersection of the D6-branes with the 16 fixed points allows us to write each twisted sector tadpole condition as a sum of 6 perfect squares

$$\begin{aligned}2\times\left[\left(\gamma_1+\gamma_3\right)^2 + \gamma_1^2 + \gamma_3^2\right] &= 0\,,\\ 2\times\left[\left(\gamma_2+\gamma_4\right)^2 + \gamma_2^2 + \gamma_4^2\right] &= 0\,. \end{aligned} \tag{3.10}$$

Apparently, in order to satisfy these 12 conditions we have to choose $\gamma_i = 0$ for all $i \in \{1,2,3,4\}$.

For the **ABB** configuration the untwisted tadpole condition becomes

$$\frac{1}{4}\left(\mathrm{M}^2 - 16\mathrm{M} + 64\right) = \frac{1}{4}\left(\mathrm{M} - 8\right)^2 = 0\,, \tag{3.11}$$

with unchanged conditions for the twisted sector tadpoles. Thus, the $\mathbb{Z}_4$ orientifold gives rise to two different models, one with gauge group of rank 16 and the other one with gauge group of rank 8.

Model	$(6_i, 6_i)$	$(6_i, 6_{i+1})$	$(6_i, 6_{i+2})$
ABA	$\mathrm{U}(8) \times \mathrm{U}(8)$ $(\mathbf{64}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{64}) \oplus$ $2(\mathbf{28}, \mathbf{1}) \oplus 2(\overline{\mathbf{28}}, \mathbf{1}) \oplus$ $2(\mathbf{1}, \mathbf{28}) \oplus 2(\mathbf{1}, \overline{\mathbf{28}})$	$(\mathbf{8}, \overline{\mathbf{8}}) \oplus (\overline{\mathbf{8}}, \mathbf{8})$	$2(\mathbf{64}, \mathbf{1}) \oplus 2(\mathbf{1}, \mathbf{64})$
ABB	$\mathrm{U}(4) \times \mathrm{U}(4)$ $(\mathbf{16}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{16}) \oplus$ $2(\mathbf{6}, \mathbf{1}) \oplus 2(\overline{\mathbf{6}}, \mathbf{1}) \oplus$ $2(\mathbf{1}, \mathbf{6}) \oplus 2(\mathbf{1}, \overline{\mathbf{6}})$	$2(\mathbf{4}, \overline{\mathbf{4}}) \oplus 2(\overline{\mathbf{4}}, \mathbf{4})$	$2(\mathbf{16}, \mathbf{1}) \oplus 2(\mathbf{1}, \mathbf{16})$

Table 3: $\mathcal{N} = 1$ open spectra for the $\mathbb{Z}_4$.

The massless closed string spectrum of the **ABA** model receives six chiral multiplets from the untwisted sector besides the graviton and dilaton multiplets. Moreover, in the $\Theta^{1,3}$ twisted sectors the 16 fixed points are all invariant under $\mathcal{R}$ and give rise to 16 chiral multiplets. Finally, the 16 fixed points of Θ^2 need to be distinguished with respect to their mapping under $\Theta, \mathcal{R}$ and even $\Theta\mathcal{R}$ providing another 15 chiral and 1 vector multiplets.

Open strings in the $(6_i, 6_i)$ sector carry eight massless states, which provide a vector multiplet in the gauge group $\mathrm{U}(8) \times \mathrm{U}(8)$, a chiral multiplet in the adjoint representation $((\mathbf{64}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{64}))$ and two chiral multiplets in the $((\mathbf{28}, \mathbf{1}) \oplus (\overline{\mathbf{28}}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{28}) \oplus (\mathbf{1}, \overline{\mathbf{28}}))$ representation. In the $(6_i, 6_{i+2})$ sector one has to take into account the extra minus signs in the loop channel Möbius strip amplitude (3.7), implying that compared to the $(6_i, 6_i)$ sector both the $\Omega\mathcal{R}$ and the Θ^2 projection change sign. Considering also the multiplicity due to the twofold degeneracy of the ground state in this sector and an extra factor of two from the intersections between 6_i and 6_{i+2} branes one gets two chiral multiplets in the $((\mathbf{64}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{64}))$ representation. Finally the $(6_i, 6_{i+1})$ strings only carry a single massless state to be counted with two orientations giving rise to one chiral multiplet in the $\big((\mathbf{8}, \overline{\mathbf{8}}) \oplus (\overline{\mathbf{8}}, \mathbf{8})\big)$ representation. Inspecting the massless open string spectrum reveals that it is not only non-chiral but surprisingly fits into $\mathcal{N} = 2$ multiplets. However, the D-branes in the $(6_i, 6_{i+1})$ sector only preserve $\mathcal{N} = 1$ supersymmetry, so that the appearance of $\mathcal{N} = 2$ multiplets is purely accidental. The complete $\mathcal{N} = 1$ supersymmetric massless spectrum is shown in the tables 2 and 3, where we have also summarized the results for the **ABB** model.

4. Results for the $\mathbb{Z}_3$ orientifol

In this section we make some remarks on the specialities of the other models we have explicitly computed. We show how the lattices and brane configurations look like, give the tadpole cancellation conditions and their solutions in terms of the gauge groups. Finally we display the massless open and closed string spectra. The details of the computation are collected in the appendix.

The $\mathbb{Z}_3$ orientifold is very similar to the six-dimensional $\mathbb{Z}_3$ model that has been discussed in some detail in [17]. As was pointed out in [19] one is free to choose any of the four possible lattices $\{\mathbf{AAA}, \mathbf{AAB}, \mathbf{ABB}, \mathbf{BBB}\}$. In all four cases the D6-branes are located in the same way on all three 2-tori as shown in figure 3. The lattice $\mathbf{A}^{3-i}\mathbf{B}^i$ on the one hand leads to an overall factor of 3^i for all the amplitudes and on the other hand causes the number of intersections to be 3^i, too. These extra factors affect the multiplicities for the different massless open string modes. In all cases the tadpole cancellation condition reads

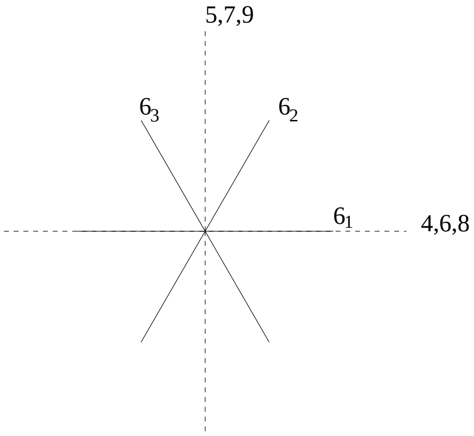

Figure 3: Branes on the $\mathbb{Z}_3$ torus.

$$\frac{1}{2}\left(\mathrm{M}^2 - 8\mathrm{M} + 16\right) = \frac{1}{2}\left(\mathrm{M} - 4\right)^2 = 0\,, \tag{4.1}$$

giving an SO(4) gauge symmetry. The $\mathcal{N} = 1$ supersymmetric spectra of all the models are collected in the tables 2 and 4.

Note, that in the $(6_i, 6_{i+1})$ sector, the phases in the definition and modular properties of the ϑ-functions lead to the symmetric representation of the gauge group.

Model	$(6_i, 6_i)$	$(6_i, 6_{i+1})$
AAA	$\mathrm{SO}(4) + 3(\mathbf{6})$	$(\mathbf{10})$
AAB	$\mathrm{SO}(4) + 3(\mathbf{6})$	$3(\mathbf{10})$
ABB	$\mathrm{SO}(4) + 3(\mathbf{6})$	$9(\mathbf{10})$
BBB	$\mathrm{SO}(4) + 3(\mathbf{6})$	$27(\mathbf{10})$

Table 4: $\mathcal{N} = 1$ open spectra for the $\mathbb{Z}_3$.

5. Results for the $\mathbb{Z}_6$ and $\mathbb{Z}_6'$ orientifolds

For each of the two $\mathbb{Z}_6$ orbifold groups we obtain two inequivalent models on the massless level. As mentioned in section 2, we are forced to choose lattices $\mathbf{AB}$ or $\mathbf{BA}$ in the two directions with Nv_i odd and can arbitrarily choose the lattice for the third $\mathbb{T}^2$ to be either $\mathbf{A}$ or $\mathbf{B}$. The two lattices of $\mathbf{A}$ and $\mathbf{B}$ type are shown in figure 4.

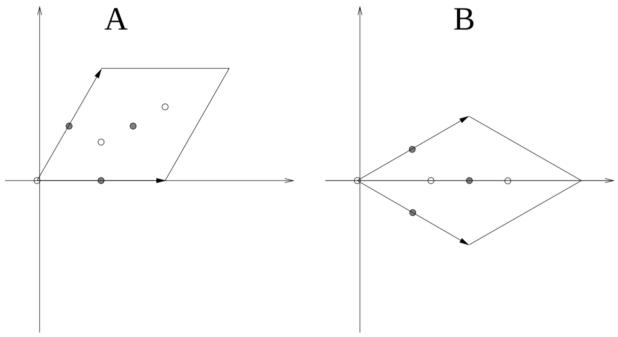

Figure 4: The $\mathbb{Z}_6$ lattices.

The location of the D6-branes is chosen according to the general rules and we depict it for the $\mathbb{Z}_6$ orientifold in figure 5.

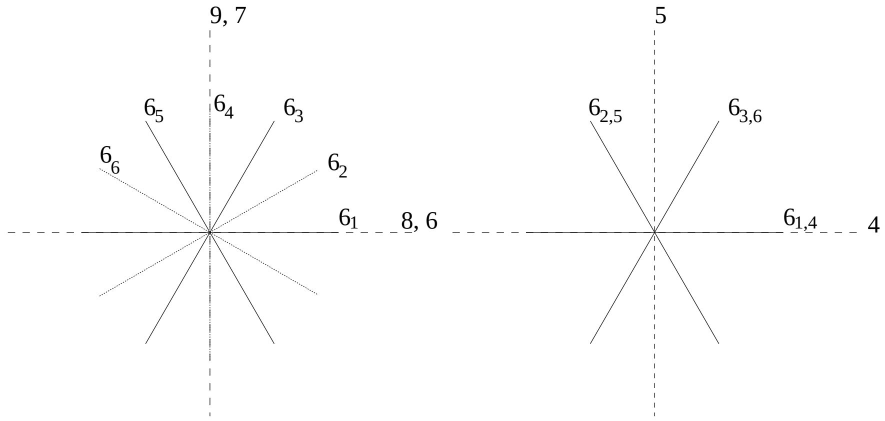

Figure 5: Branes on the $\mathbb{Z}_6$ torus.

Model	$(6_i, 6_i)$	$(6_i, 6_{i+1})$	$(6_i, 6_{i+2})$	$(6_i, 6_{i+3})$
ABA	$U(2) \times U(2)$ $(\mathbf{4}, \mathbf{1_0}) \oplus (\mathbf{1_0}, \mathbf{4}) \oplus$ $2(\mathbf{1}, \mathbf{1_0}) \oplus 2(\overline{\mathbf{1}}, \mathbf{1_0}) \oplus$ $2(\mathbf{1_0}, \mathbf{1}) \oplus 2(\mathbf{1_0}, \overline{\mathbf{1}})$	$(\mathbf{2}, \overline{\mathbf{2}}) \oplus (\overline{\mathbf{2}}, \mathbf{2})$	$(\mathbf{3}, \mathbf{1_0}) \oplus (\overline{\mathbf{3}}, \mathbf{1_0}) \oplus$ $(\mathbf{1_0}, \mathbf{3}) \oplus (\mathbf{1_0}, \overline{\mathbf{3}}) \oplus$ $2(\mathbf{4}, \mathbf{1_0}) \oplus 2(\mathbf{1_0}, \mathbf{4})$	$4(\mathbf{2}, \overline{\mathbf{2}}) \oplus 4(\overline{\mathbf{2}}, \mathbf{2})$
ABB	$U(2) \times U(2)$ $(\mathbf{4}, \mathbf{1_0}) \oplus (\mathbf{1_0}, \mathbf{4}) \oplus$ $2(\mathbf{1}, \mathbf{1_0}) \oplus 2(\overline{\mathbf{1}}, \mathbf{1_0}) \oplus$ $2(\mathbf{1_0}, \mathbf{1}) \oplus 2(\mathbf{1_0}, \overline{\mathbf{1}})$	$3(\mathbf{2}, \overline{\mathbf{2}}) \oplus 3(\overline{\mathbf{2}}, \mathbf{2})$	$3(\mathbf{3}, \mathbf{1_0}) \oplus 3(\overline{\mathbf{3}}, \mathbf{1_0}) \oplus$ $3(\mathbf{1_0}, \mathbf{3}) \oplus 3(\mathbf{1_0}, \overline{\mathbf{3}}) \oplus$ $6(\mathbf{4}, \mathbf{1_0}) \oplus 6(\mathbf{1_0}, \mathbf{4})$	$4(\mathbf{2}, \overline{\mathbf{2}}) \oplus 4(\overline{\mathbf{2}}, \mathbf{2})$

Table 5: $\mathcal{N} = 1$ open spectra for the $\mathbb{Z}_6$.

The untwisted tadpole cancellation condition in this case is

$$\frac{1}{2}\left(\mathrm{M}^2 - 8\mathrm{M} + 16\right) = \frac{1}{2}\left(\mathrm{M} - 4\right)^2 = 0\,. \tag{5.1}$$

The $\mathbb{Z}_2$ twisted tadpole condition is similar to the one in (3.10) and can be satisfied by choosing traceless $\gamma_3^{(i)}$ matrices, leading to a $U(2) \times U(2)$ gauge group. Taking all twisted sectors and intersection points into account we derive the $\mathcal{N} = 1$ supersymmetric open string massless spectrum shown in table 5. To distinguish $SU(2)$ singlets which are charged or neutral under the $U(1)$ we use the notation $\mathbf{1}$ and $\mathbf{1_0}$. The $\mathbf{2}$ and $\overline{\mathbf{2}}$ are similarly distinguished by their abelian $U(1)$ charges.

Finally we discuss the $\mathbb{Z}_6'$ orientifold where one has to choose the D6-branes as shown in figure 6.

Special care needs to be taken in the Θ^2-twisted sector of the $\mathbb{Z}_6'$ model, as the partition function of the KK and W states as well as the number of fixed points and intersections does depend on the factor of the gauge group. Eventually, everything comes out just right and leads to the complete projector in the tree channel. As a consequence the two choices **AB** and **BA** in the directions of odd Nv_i lead to slightly different models. As can be seen by direct computation, on the massless level this difference is only an exchange of the two gauge factors.

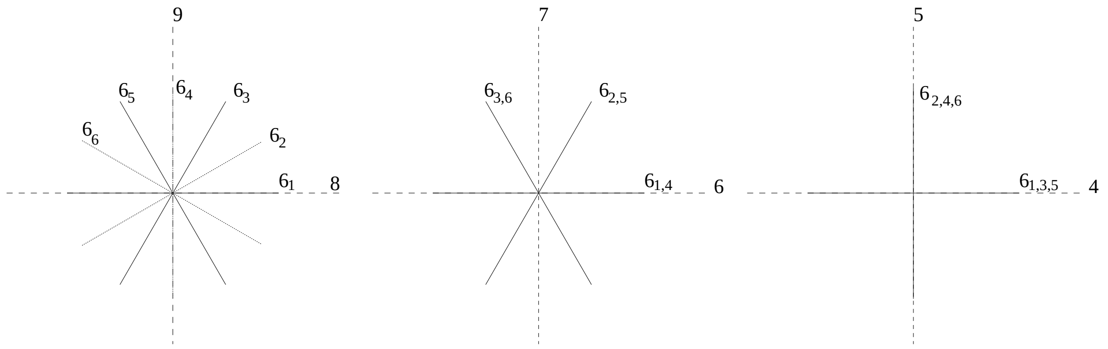

Figure 6: Branes on the $\mathbb{Z}_6'$ torus.

Model	$(6_i, 6_i)$	$(6_i, 6_{i+1})$	$(6_i, 6_{i+2})$	$(6_i, 6_{i+3})$
AAB	$\mathrm{U}(2) \times \mathrm{U}(2)$ $(\mathbf{4}, \mathbf{1_0}) \oplus (\mathbf{1_0}, \mathbf{4}) \oplus$ $2(\mathbf{1}, \mathbf{1_0}) \oplus 2(\overline{\mathbf{1}}, \mathbf{1_0}) \oplus$ $2(\mathbf{1_0}, \mathbf{1}) \oplus 2(\mathbf{1_0}, \overline{\mathbf{1}})$	$2(\mathbf{2}, \overline{\mathbf{2}}) \oplus 2(\overline{\mathbf{2}}, \mathbf{2})$	$(\mathbf{1}, \mathbf{1_0}) \oplus (\overline{\mathbf{1}}, \mathbf{1_0}) \oplus$ $3(\mathbf{1_0}, \mathbf{1}) \oplus 3(\mathbf{1_0}, \overline{\mathbf{1}}) \oplus$ $(\mathbf{4}, \mathbf{1_0}) \oplus 3(\mathbf{1_0}, \mathbf{4})$	$4(\mathbf{2}, \overline{\mathbf{2}}) \oplus 4(\overline{\mathbf{2}}, \mathbf{2})$
ABB	$\mathrm{U}(2) \times \mathrm{U}(2)$ $(\mathbf{4}, \mathbf{1_0}) \oplus (\mathbf{1_0}, \mathbf{4}) \oplus$ $2(\mathbf{1}, \mathbf{1_0}) \oplus 2(\overline{\mathbf{1}}, \mathbf{1_0}) \oplus$ $2(\mathbf{1_0}, \mathbf{1}) \oplus 2(\mathbf{1_0}, \overline{\mathbf{1}})$	$6(\mathbf{2}, \overline{\mathbf{2}}) \oplus 6(\overline{\mathbf{2}}, \mathbf{2})$	$3(\mathbf{1}, \mathbf{1_0}) \oplus 3(\overline{\mathbf{1}}, \mathbf{1_0}) \oplus$ $9(\mathbf{1_0}, \mathbf{1}) \oplus 9(\mathbf{1_0}, \overline{\mathbf{1}}) \oplus$ $3(\mathbf{4}, \mathbf{1_0}) \oplus 9(\mathbf{1_0}, \mathbf{4})$	$4(\mathbf{2}, \overline{\mathbf{2}}) \oplus 4(\overline{\mathbf{2}}, \mathbf{2})$

Table 6: $\mathcal{N} = 1$ open spectra for the $\mathbb{Z}_6'$.

Again the untwisted tadpole cancellation condition is unaffected by the change and reads explicitly:

$$\frac{1}{2}\left(\mathrm{M}^2 - 8\mathrm{M} + 16\right) = \frac{1}{2}\left(\mathrm{M} - 4\right)^2 = 0\,. \tag{5.2}$$

The gauge group again is $\mathrm{U}(2) \times \mathrm{U}(2)$ and the $\mathcal{N} = 1$ supersymmetric massless open string spectrum is shown in table 6.

We used the first gauge factor for the theory on the odd $D6_i$-branes. In contrast to the other cases discussed so far, the massless spectrum of the $\mathbb{Z}_6'$ orientifold is not invariant under exchange of the two $\mathrm{U}(2)$ gauge factors.

6. Conclusions

In this paper we have studied $\Omega\mathcal{R}$ orientifolds of type IIA in four dimensions. We found a couple of models for which the tadpole cancellation conditions could be satisfied, leading to non-chiral massless spectra on the D6-branes. It would be interesting to study the other $\mathcal{N} = 1$ supersymmetric orbifolds from table 1, as well. However, for these cases the contribution of the lattices is more involved, as they cannot be factorized as $\mathbb{T}^6 = \mathbb{T}^2 \times \mathbb{T}^2 \times \mathbb{T}^2$. Even knowing the basis vectors of $\mathbb{T}^6$ tori with $\mathbb{Z}_7$, $\mathbb{Z}_8$ and $\mathbb{Z}_{12}$ symmetry is not enough, one also has to orientate the lattice in such a way that also the reflection $\mathcal{R}$ acts crystallograhically, which may give rise to a variety of distinct models.

JHEP01(2000)040

As we pointed out in [17] these $\Omega\mathcal{R}$ orientifolds might also be interesting for non-supersymmetric generalizations. In particular, the $\Omega(-1)^{F_L}$ orientifolds of type 0B are known to be sometimes free of tachyons leading to classically stable non-supersymmetric string vacua [25]. However, for standard $\mathbb{Z}_N$ orientifolds with $N \geq 3$ some tachyons appearing in the twisted sectors survive the Ω projection. The reason is, that Ω exchanges the Θ^k twisted sector with the Θ^{-k} twisted sector, so that one linear combination of both tachyons survives. Since in the $\Omega\mathcal{R}$ case, the twisted sectors are left invariant one may hope that at least the closed string sector of type 0A $\mathbb{Z}_N$ orientifolds is free of tachyons. Sometimes this is indeed the case. However, it is easy to see that the tachyons now return in the open string sectors, namely in the twisted open string sectors. Thus, in contrast to our earlier expectation type 0A $\Omega\mathcal{R}(-1)^{F_L}$ orientifolds also generically contain open string tachyons and do not behave better than standard type 0B $\Omega(-1)^{F_L}$ orientifolds.

Acknowledgments

We would like to thank A. Kumar, who was involved in early stages of this work, as well as M. Gaberdiel, D. Lüst, A. Miemiec and H. Skarke for encouraging discussions and helpful remarks.

A. One-loop partition functions

The general strategy to compute the several one loop contributions to the massless tadpoles is as usual: We compute the loop diagrams of the Klein bottle, the annulus and the Möbius strip and convert the results via a modular S transformation sending $t \mapsto 1/t$ into the tree channel, expand the integrand for large distances and extract the coefficients of the divergent terms. This gives us a condition for the number of branes and for the action of non-trivially represented Chan-Paton matrices, the only one being $\gamma^{(i)}_{N/2}$ for the $\mathbb{Z}_N$ orbifolds with even N. In general we denote the unitary action on Chan-Paton labels by $\gamma^{(i)}_k$ and $\gamma^{(i)}_{\Omega\mathcal{R}k}$ for Θ^k and $\Omega\mathcal{R}\Theta^k$ acting on the D6$_i$-branes respectively.

A.1 Klein bottle

The Klein bottle amplitude in the loop channel has the general form

$$\mathcal{K} = 4\frac{V_4}{(8\pi^2\alpha')^2}\int_0^\infty \frac{dt}{t^3}\left(\frac{1}{4N}\sum_{n,k=0}^{N}\mathcal{K}^{(n,k)}\mathcal{L}^{(n,k)}_{\mathcal{K}}\right), \tag{A.1}$$

with $\mathcal{K}^{(n,k)}$ denoting the trace over oscillators (osc) in the sector twisted by Θ^n with the insertion of Θ^k inside

$$\mathcal{K}^{(n,k)} \equiv \mathrm{Tr}^{(n)}_{\mathrm{osc}}\left(\Omega\mathcal{R}\Theta^k\left(1+(-1)^F\right)e^{-2\pi t\left(L_0+\bar{L}_0\right)}\right) \tag{A.2}$$

and $\mathcal{L}_{\mathcal{K}}^{(n,k)}$ standing for the trace over bosonic zero modes, quantized momenta and windings (KK+W), i.e.

$$\mathcal{L}_{\mathcal{K}}^{(n,k)} \equiv \chi_{\mathcal{K}}^{(n,k)} \, \mathrm{Tr}_{\mathrm{KK+W}}^{(n)} \left(\Omega \mathcal{R} \Theta^k \; e^{-2\pi t \left(L_0 + \bar{L}_0 \right)} \right). \tag{A.3}$$

Here $\chi_{\mathcal{K}}^{(n,k)}$ is the number of fixed points of Θ^n, which are invariant under the operator $\mathcal{R}\Theta^k$ in the trace. Except for the $\mathbb{Z}_6'$ orbifold these numbers, as well as the entire traces are equal for any insertion of Θ^k and even in that case only the lattice contributions differ, so we can factorize the partition function as in (A.1). In all other cases we omit the superscript k, writing $\mathcal{L}_{\mathcal{K}}^{(n)}$, $\mathcal{K}^{(n)}$ and $\chi_{\mathcal{K}}^{(n)}$. As a shorthand for the contributions of the bosonic zero modes we use

$$\mathcal{L}\left[\alpha, \beta\right] \equiv \left(\sum_{m \in \mathbb{Z}} e^{-\alpha \pi t m^2 / R^2} \right) \left(\sum_{n \in \mathbb{Z}} e^{-\beta \pi t n^2 R^2} \right). \tag{A.4}$$

For all amplitudes we used the convention

$$\sum_{(nv_i, kv_i) \notin \mathbb{Z}^2} nv_i = 0 \, . \tag{A.5}$$

The oscillator sums lead to the generic expression

$$\mathcal{K}_{(v_1,v_2,v_3)}^{(n)} \equiv (1-1) \, \frac{\vartheta \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}}{\eta^3} \prod_{nv_i \notin \mathbb{Z}} \left(\frac{\vartheta \begin{bmatrix} nv_i \\ 1/2 \end{bmatrix}}{\vartheta \begin{bmatrix} 1/2 + nv_i \\ 1/2 \end{bmatrix}} e^{\pi i \langle nv_i \rangle} \right) \prod_{nv_i \in \mathbb{Z}} \left(\frac{\vartheta \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}}{\eta^3} \right), \tag{A.6}$$

with argument $q = \exp\left(-4\pi t\right)$ and

$$\langle nv_i \rangle \equiv nv_i - [nv_i] - \frac{1}{2} \, . \tag{A.7}$$

$\mathcal{L}_{\mathcal{K}}^{(n,k)}$ has contributions whenever $nv_i \in \mathbb{Z}$. In contrast to the oscillator part we cannot give any generic formula for the lattice sums, since one has to take into account which momentum and winding states are invariant under the respective operator in the trace. This does not only depend on the orbifold group action, as given by the v_i, but also on the orientation of the lattice with respect to the reflection $\mathcal{R}$. For the tree channel amplitudes we also define appropriate abbreviations:

$$\begin{aligned} \tilde{\mathcal{K}}_{(v_1,v_2,v_3)}^{(n)} &\equiv (1-1) \, \frac{\vartheta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}}{\eta^3} \prod_{nv_i \notin \mathbb{Z}} \left(\frac{\vartheta \begin{bmatrix} 1/2 \\ nv_i \end{bmatrix}}{\vartheta \begin{bmatrix} 1/2 \\ 1/2 + nv_i \end{bmatrix}} \right) \prod_{nv_i \in \mathbb{Z}} \left(\frac{\vartheta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}}{\eta^3} \right), \\ \tilde{\mathcal{L}}\left[\alpha, \beta\right] &\equiv \left(\sum_{m \in \mathbb{Z}} e^{-\alpha \pi l m^2 R^2} \right) \left(\sum_{n \in \mathbb{Z}} e^{-\beta \pi l n^2 / R^2} \right) \end{aligned} \tag{A.8}$$

the argument of $\tilde{\mathcal{K}}^{(n)}_{(v_1,v_2,v_3)}$ being $\tilde{q} = \exp(-4\pi l)$. The expected prefactor to yield the complete projector is therefore

$$\prod_{nv_i \notin \mathbb{Z}} \left(-2\sin(\pi n v_i)\right). \tag{A.9}$$

The volume factors from the modular transformation formally cancel out, the divergence being related to a tadpole of a 7-form field.

A.2 Annulus

The open string diagrams can also be factorized into oscillator and momentum and winding parts. For the annulus the loop channel reads

$$\mathcal{A} = \frac{V_4}{(8\pi^2\alpha')^2} \int_0^\infty \frac{dt}{t^3} \left(\frac{1}{4N} \sum_{n,k,i=0}^{N} \mathrm{tr}\left(\gamma_k^{(i)}\right) \mathrm{tr}\left(\left(\gamma_k^{(i+n)}\right)^{-1}\right) \mathcal{A}^{(n,k)}\, \mathcal{L}_{\mathcal{A}}^{(n,k,i)} \right) \tag{A.10}$$

and analogously to the above we define

$$\mathcal{A}^{(n,k)} \equiv \mathrm{Tr}^{(i,i+n)}_{\mathrm{osc}} \left(\Theta^k \left(1 + (-1)^F\right) e^{-2\pi t L_0} \right), \tag{A.11}$$

where the trace is to be performed over the oscillator excitations of the open strings stretching between the i and $i+n$ branes, with only $k \in \{0, N/2\}$ contributions non-vanishing. Concerning the Chan-Paton matrices, we choose $\mathrm{tr}(\gamma_0^{(i)})^2 = \mathrm{M}^2$, leaving only $\mathrm{tr}(\gamma_{N/2}^{(i)})$ and the number of branes, M, to be determined by the tadpole cancellation conditions. The bosonic zero modes contribute

$$\mathcal{L}_{\mathcal{A}}^{(n,k,\pm)} \equiv \chi_{\mathcal{A}}^{(n,k,\pm)}\, \mathrm{Tr}^{(i,i+n)}_{\mathrm{KK+W}} \left(\Theta^k\, e^{-2\pi t L_0} \right). \tag{A.12}$$

$\chi_{\mathcal{A}}^{(n,k,\pm)}$ is the number of intersections of the two types of branes on the torus, which are invariant under Θ^k. The trace (A.12) is only different from 1 if the i and $i+n$ branes coincide in some (real) direction on the torus $\mathbb{T}^6$ and Θ^k acts trivially in the complex plane of this direction. In the $\mathbb{Z}_6'$ orbifold there is another distinction between odd and even i, which we have reserved the extra superscript $\pm$ for. For the oscillator part we get the generic formula

$$\mathcal{A}^{(n,k)}_{(v_1,v_2,v_3)} \equiv (1-1)\, \frac{\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}}{\eta^3} \prod_{(nv_i,kv_i)\notin\mathbb{Z}^2} \left(\frac{2^\delta \vartheta\begin{bmatrix}nv_i\\1/2+kv_i\end{bmatrix}}{\vartheta\begin{bmatrix}1/2+nv_i\\1/2+kv_i\end{bmatrix}} e^{\pi i \langle n v_i\rangle} \right) \prod_{(nv_i,kv_i)\in\mathbb{Z}^2} \left(\frac{\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}}{\eta^3} \right), \tag{A.13}$$

with argument $q = \exp(-2\pi t)$. The second product is empty, except if both nv_i and kv_i are integers, which is also the only case, in which there are contributions to $\mathcal{L}_{\mathcal{A}}^{(n,k,\pm)}$. We have introduced factors of 2^δ for cancelling inappropriate factors occurring in the theta functions by defining $\delta = 1$ if $nv_i \in \mathbb{Z}$ and $kv_i \in \mathbb{Z} + 1/2$ and $\delta = 0$ otherwise. One could have omitted these extra factors by using the notations of [14], for instance, but this would then imply a definition of the case $\delta = 1$ via some limit of the otherwise undefined formula, which we preferred to avoid. Again there is no generic expression for the lattice sums and one needs to consider not only the type of brane as given by i, n but also their orientation on the torus, in order to determine the normalization of momenta and winding states. $\mathcal{L}_{\mathcal{A}}^{(n,k,\pm)}$ can then be written in the form of (A.4), of course. For the tree channel oscillators we use

$$\tilde{\mathcal{A}}^{(n,k)}_{(v_1,v_2,v_3)} \equiv (1-1)\, \frac{\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}}{\eta^3} \prod_{(nv_i,kv_i)\notin\mathbb{Z}^2} \left(\frac{\vartheta\begin{bmatrix}1/2+kv_i\\nv_i\end{bmatrix}}{\vartheta\begin{bmatrix}1/2+kv_i\\1/2+nv_i\end{bmatrix}} \right) \prod_{(nv_i,kv_i)\in\mathbb{Z}^2} \left(\frac{\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}}{\eta^3} \right), \quad \text{(A.14)}$$

with argument $\tilde{q} = \exp(-4\pi l)$. In the tree channel the annulus only contributes to the untwisted and $\Theta^{N/2}$ twisted sector, corresponding to the 7-form and a twisted 5-form tadpole.

A.3 Möbius strip

Finally we need to go through the Möbius strip amplitude. The loop channel expression is

$$\mathcal{M} = -\frac{V_4}{(8\pi^2\alpha')^2} \int_0^\infty \frac{dt}{t^3} \left(\frac{1}{4N} \sum_{n,k,i=0}^{N} \mathrm{tr}\left(\left(\gamma^{(i)}_{\Omega\mathcal{R}k}\right)^{-1} \left(\gamma^{(i)}_{\Omega\mathcal{R}k}\right)^{\mathrm{T}} \right) \mathcal{M}^{(n,k)}\, \mathcal{L}_{\mathcal{M}}^{(n,k,i)} \right), \quad \text{(A.15)}$$

with the oscillator

$$\mathcal{M}^{(n,k)} \equiv \mathrm{Tr}^{(1,1+n)}_{\mathrm{osc}} \left(\Omega\mathcal{R}\Theta^k \left(1 + (-1)^F\right) e^{-2\pi t L_0} \right) \quad \text{(A.16)}$$

and the zero mode trace

$$\mathcal{L}_{\mathcal{M}}^{(n,k,\pm)} \equiv \chi_{\mathcal{M}}^{(n,k,\pm)}\ \mathrm{Tr}^{(i,i+n)}_{\mathrm{KK+W}} \left(\Omega\mathcal{R}\Theta^k\, e^{-2\pi t L_0} \right). \quad \text{(A.17)}$$

Now $\chi_{\mathcal{M}}^{(n,k,\pm)}$ denotes the number of intersection points of the i and $i+n$ branes, invariant under $\mathcal{R}\Theta^k$. By looking at the following chain of mappings of open strings

$$\begin{aligned} (i, i+n) &\xrightarrow{\Theta^k} (i+2k, i+n+2k) \quad \text{(A.18)} \\ &\xrightarrow{\mathcal{R}} (2-i-2k, 2-i-n-2k) \xrightarrow{\Omega} (2-i-n-2k, 2-i-2k) \end{aligned}$$

one realizes that only strings that satisfy $2(k+i-1)+n = 0 \bmod N$ can contribute in the Möbius strip. If N is even, the relation has two solutions $k \equiv 1-i-n/2, 1-i-n/2+N/2$ for any combination of $i, n/2 \in \mathbb{Z}$ and only one for any $i, n \in \mathbb{Z}$, if N is odd. By regarding

$$\Omega\mathcal{R}\Theta^{1-i-n/2} = \Theta^{-(1-i)/2} \left(\Omega\mathcal{R}\Theta^{-n/2} \right) \Theta^{(1-i)/2} \quad \text{(A.19)}$$

one finds that $\Omega\mathcal{R}\Theta^{1-i-n/2}$ leaves the $(6_i, 6_{i+n})$ strings invariant just like $\Omega\mathcal{R}\Theta^{-n/2}$ the $(6_1, 6_{1+n})$ strings, which are those that start on the brane in the fixed plane of

$\mathcal{R}$. On these $\Omega\mathcal{R}$ itself acts simply like the ordinary Ω on strings with Neumann boundary conditions only, as the additional signs of Ω cancel the reflection. The oscillator traces for all other values of i are identical to the $i = 1$ case, but the contributions arise from different n, k combinations. For the oscillator part we then get the formula

$$\mathcal{M}^{(n,k)}_{(v_1,v_2,v_3)} \equiv (1-1)\,\frac{\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}}{\eta^3}\prod_{(nv_i,kv_i)\notin\mathbb{Z}^2}\left(\frac{2^\delta\vartheta\begin{bmatrix}1/2+nv_i\\kv_i\end{bmatrix}}{\vartheta\begin{bmatrix}1/2+nv_i\\1/2+kv_i\end{bmatrix}}e^{\pi i\langle nv_i\rangle}\right)\prod_{(nv_i,kv_i)\in\mathbb{Z}^2}\left(\frac{\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}}{\eta^3}\right), \tag{A.20}$$

with argument $q = -\exp(-2\pi t)$, which enforces a different modular transformation property. As for the annulus, there are lattice contributions if nv_i and kv_i are both integers. They differ from those of the annulus, as one needs to sum over states invariant under $\Omega\mathcal{R}$, which boils down to doubling the winding quantum numbers, except for the $\mathbb{Z}_4$, where they remain unchanged for the **A** type of lattice, while they are still doubled for the **B** type. For the modular transformed Möbius strip we use

$$\tilde{\mathcal{M}}^{(m)}_{(v_1,v_2,v_3)} \equiv (1-1)\,\frac{\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}}{\eta^3}\prod_{mv_i\notin\mathbb{Z}}\left(\frac{\vartheta\begin{bmatrix}1/2\\mv_i\end{bmatrix}}{\vartheta\begin{bmatrix}1/2\\1/2+mv_i\end{bmatrix}}\right)\prod_{mv_i\in\mathbb{Z}}\left(\frac{\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}}{\eta^3}\right), \tag{A.21}$$

with argument $\tilde{q} = -\exp(-4\pi l)$.

B. Definitions and modular transformation formulas

In this appendix we give basic definitions and fix some notation. We frequently employ the Jacobi theta function and Dedekind eta function

$$\begin{aligned}\vartheta\begin{bmatrix}\alpha\\\beta\end{bmatrix}(t) &= \sum_{n\in\mathbb{Z}} q^{(n+\alpha)^2/2}\; e^{2\pi i(n+\alpha)\beta}\,,\\ \eta(t) &= q^{1/24}\prod_{n=1}^{\infty}(1-q^n)\,,\end{aligned} \tag{B.1}$$

setting $q \equiv e^{-2\pi t}$. The argument α is defined modulo $\mathbb{Z}$ and in order to directly use the product expansion

$$\frac{\vartheta\begin{bmatrix}\alpha\\\beta\end{bmatrix}}{\eta}(t) = e^{2\pi i\alpha\beta}\; q^{\alpha^2/2-1/24}\prod_{n=1}^{\infty}\left(\left(1+q^{n-1/2+\alpha}e^{2\pi i\beta}\right)\left(1+q^{n-1/2-\alpha}e^{-2\pi i\beta}\right)\right) \tag{B.2}$$

one needs to choose $\alpha \in (-1/2, 1/2]$.

The modular S transformation is

$$\begin{aligned}\vartheta\begin{bmatrix}\alpha\\\beta\end{bmatrix}\left(t^{-1}\right) &= \sqrt{t}\,e^{2\pi i\alpha\beta}\vartheta\begin{bmatrix}-\beta\\\alpha\end{bmatrix}(t)\,,\\ \eta\left(t^{-1}\right) &= \sqrt{t}\,\eta(t)\,.\end{aligned} \tag{B.3}$$

In order to rewrite the loop channel Möbius strip amplitude in terms of the appropriate tree level fields we use the identity

$$\frac{\vartheta\begin{bmatrix}\alpha+1/2\\ \beta\end{bmatrix}}{\vartheta\begin{bmatrix}\alpha+1/2\\ \beta+1/2\end{bmatrix}}(-q) = e^{-\pi i\alpha}\frac{\vartheta\begin{bmatrix}(\alpha+1)/2\\ \alpha/2+\beta\end{bmatrix}\vartheta\begin{bmatrix}\alpha/2\\ (\alpha+1)/2+\beta\end{bmatrix}}{\vartheta\begin{bmatrix}(\alpha+1)/2\\ (\alpha+1)/2+\beta\end{bmatrix}\vartheta\begin{bmatrix}\alpha/2\\ \alpha/2+\beta\end{bmatrix}}\left(q^2\right) \tag{B.4}$$

for $-1 < \alpha \leq 0$. For the modular transformation of lattice and momentum and winding sums we need the Poisson resummation formula

$$\sum_{n\in\mathbb{Z}} e^{-\pi n^2/t} = \sqrt{t}\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}\,. \tag{B.5}$$

References

[1] I. Antoniadis, *A possible new dimension at a few TeV*, *Phys. Lett.* **B 246** (1990) 377;
J.D. Lykken, *Weak scale superstrings*, *Phys. Rev.* **D 54** (1996) 3693 [hep-th/9603133];
N. Arkani-Hamed, S. Dimopoulos and G. Dvali, *The hierarchy problem and new dimensions at a millimeter*, *Phys. Lett.* **B 429** (1998) 263 [hep-ph/9803315]; *Phenomenology, astrophysics and cosmology of theories with sub-millimeter dimensions and TeV scale quantum gravity*, *Phys. Rev.* **D 59** (1999) 086004 [hep-ph/9807344];
I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, *New dimensions at a millimeter to a fermi and superstrings at a TeV*, *Phys. Lett.* **B 436** (1998) 257 [hep-ph/9804398];
G. Shiu and S.H.H. Tye, *TeV scale superstring and extra dimensions*, *Phys. Rev.* **D 58** (1998) 106007 [hep-th/9805157].

[2] M. Bianchi and A. Sagnotti, *On the systematics of open string theories*, *Phys. Lett.* **B 247** (1990) 517.

[3] M. Bianchi, G. Pradisi and A. Sagnotti, *Toroidal compactification and symmetry breaking in open string theories*, *Nucl. Phys.* **B 376** (1992) 365;
M. Bianchi, *A note on toroidal compactifications of the type-I superstring and other superstring vacuum configurations with 16 supercharges*, *Nucl. Phys.* **B 528** (1998) 73 [hep-th/9711201];
E. Witten, *Toroidal compactification without vector structure*, *J. High Energy Phys.* **02** (1998) 006 [hep-th/9712028];
C. Angelantonj, *Comments on open-string orbifolds with a non-vanishing B_{ab}*, hep-th/9908064.

[4] E.G. Gimon and J. Polchinski, *Consistency conditions for orientifolds and D-manifolds*, *Phys. Rev.* **D 54** (1996) 1667 [hep-th/9601038].

[5] A. Dabholkar and J. Park, *An orientifold of type-IIB theory on $K3$*, *Nucl. Phys.* **B 472** (1996) 207 [hep-th/9602030]; *Strings on orientifolds*, *Nucl. Phys.* **B 477** (1996) 701 [hep-th/9604178].

[6] E.G. Gimon and C.V. Johnson, *K3 orientifolds*, *Nucl. Phys.* **B 477** (1996) 715 [hep-th/9604129].

[7] M. Berkooz and R.G. Leigh, *A $D=4$ $N=1$ orbifold of type-I strings*, *Nucl. Phys.* **B 483** (1997) 187 [hep-th/9605049];
M. Berkooz et al., *Anomalies, dualities and topology of $D=6$ $N=1$ superstring vacua*, *Nucl. Phys.* **B 475** (1996) 115 [hep-th/9605184].

[8] C. Angelantonj, M. Bianchi, G. Pradisi, A. Sagnotti and Y.S. Stanev, *Chiral asymmetry in four-dimensional open-string vacua*, *Phys. Lett.* **B 385** (1996) 96 [hep-th/9606169].

[9] C. Angelantonj, M. Bianchi, G. Pradisi, A. Sagnotti and Y.S. Stanev, *Comments on Gepner models and type-I vacua in string theory*, *Phys. Lett.* **B 387** (1996) 743 [hep-th/9607229].

[10] R. Gopakumar and S. Mukhi, *Orbifold and orientifold compactifications of F-theory and M-theory to six and four dimensions*, *Nucl. Phys.* **B 479** (1996) 260 [hep-th/9607057].

[11] J.D. Blum and A. Zaffaroni, *An orientifold from F-theory*, *Phys. Lett.* **B 387** (1996) 71 [hep-th/9607019];
J.D. Blum, *F-theory orientifolds, M-theory orientifolds and twisted strings*, *Nucl. Phys.* **B 486** (1997) 34 [hep-th/9608053].

[12] Z. Kakushadze, *Aspects of $N=1$ type-I-heterotic duality in four dimensions*, *Nucl. Phys.* **B 512** (1998) 221 [hep-th/9704059];
Z. Kakushadze and G. Shiu, *A chiral $N=1$ type-I vacuum in four dimensions and its heterotic dual*, *Phys. Rev.* **D 56** (1997) 3686 [hep-th/9705163]; *4D chiral $N=1$ type-I vacua with and without D5-branes*, *Nucl. Phys.* **B 520** (1998) 75 [hep-th/9706051];
Z. Kakushadze, *On four dimensional $N=1$ type-I compactifications*, *Nucl. Phys.* **B 535** (1998) 311 [hep-th/9806008].

[13] G. Zwart, *Four-dimensional $N=1$ $\mathbb{Z}_N \times \mathbb{Z}_M$ orientifolds*, *Nucl. Phys.* **B 526** (1998) 378 [hep-th/9708040].

[14] G. Aldazabal, A. Font, L.E. Ibáñez and G. Violero, *$D=4$, $N=1$, type-IIB orientifolds*, *Nucl. Phys.* **B 536** (1998) 29 [hep-th/9804026];
L.E. Ibáñez, R. Rabadan and A.M. Uranga, *Anomalous* U(1)*'s in type-I and type-IIB $D=4$, $N=1$ string vacua*, *Nucl. Phys.* **B 542** (1999) 112 [hep-th/9808139];
G. Aldazabal, D. Badagnani, L.E. Ibáñez and A.M. Uranga, *Tadpole versus anomaly cancellation in $D=4$, 6 compact IIB orientifolds*, *J. High Energy Phys.* **06** (1999) 031 [hep-th/9904071].

[15] R. Blumenhagen and A. Wisskirchen, *Spectra of 4D, $N=1$ type-I string vacua on non-toroidal CY threefolds*, *Phys. Lett.* **B 438** (1998) 52 [hep-th/9806131].

[16] Z. Kakushadze, G. Shiu and S.H. H. Tye, *Type IIB orientifolds, F-theory, type-I strings on orbifolds and type-I heterotic duality*, *Nucl. Phys.* **B 533** (1998) 25 [hep-th/9804092];
Z. Kakushadze, *Non-perturbative orientifolds*, *Phys. Lett.* **B 455** (1999) 120 [hep-th/9904007]; *Type-I on (generalized) Voisin-Borcea orbifolds and non-perturbative orientifolds*, hep-th/9904211; *Non-perturbative K3 orientifolds with NS-NS B-flux*, *Phys. Lett.* **B 459** (1999) 497 [hep-th/9905033].

[17] R. Blumenhagen, L. Görlich and B. Körs, *Supersymmetric orientifolds in 6D with D-branes at angles*, hep-th/9908130.

[18] M. Bianchi, J.F. Morales and G. Pradisi, *Discrete torsion in non-geometric orbifolds and their open-string descendants*, hep-th/9910228.

[19] C. Angelantonj and R. Blumenhagen, *Discrete deformations in type-I vacua*, hep-th/9911190.

[20] C. Angelantonj, I. Antoniadis, G. D'Appollonio, E. Dudas and A. Sagnotti, *Type-I vacua with brane supersymmetry breaking*, hep-th/9911081.

[21] L. Dixon, J.A. Harvey, C. Vafa and E. Witten, *Strings on orbifolds*, *Nucl. Phys.* **B 261** (1985) 678.

[22] L. Dixon, J.A. Harvey, C. Vafa and E. Witten, *Strings on orbifolds, 2*, *Nucl. Phys.* **B 274** (1986) 285.

[23] D.G. Markushevich, M.A. Olshanetsky and A.M. Perelomov, *Description of a class of superstring compactifications related to semi-simple Lie algebras*, *Comm. Math. Phys.* **111** (1987) 247.

[24] G. Pradisi, *Type-I vacua from diagonal Z_3-orbifolds*, hep-th/9912218.

[25] A. Sagnotti, *Some properties of open string theories*, hep-th/9509080; *Surprises in open-string perturbation theory*, *Nucl. Phys.* **56B** *(Proc. Suppl.)* (1997) 332 [hep-th/9702093];
C. Angelantonj, *Non-tachyonic open descendants of the 0B string theory*, *Phys. Lett.* **B 444** (1998) 309 [hep-th/9810214];
R. Blumenhagen, A. Font and D. Lüst, *Tachyon-free orientifolds of type-0B strings in various dimensions*, *Nucl. Phys.* **B 558** (1999) 159 [hep-th/9904069];
R. Blumenhagen and A. Kumar, *A note on orientifolds and dualities of type-0B string theory*, *Phys. Lett.* **B 464** (1999) 46 [hep-th/9906234];
K. Förger, *On non-tachyonic $\mathbb{Z}_N \times \mathbb{Z}_M$ orientifolds of type-0B string theory*, *Phys. Lett.* **B 469** (1999) 113 [hep-th/9909010].

JHEP01(2000)040